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Stability analysis for Lotka–Volterra systems based on an algorithm of real root isolation☆

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Abstract

Based on the theory of monotone flows of solutions of systems of differential equations, the Routh–Hurwitz theorem and a real root isolation algorithm of multivariate polynomials are applied to a class of Lotka–Volterra diffusion systems. An algorithm to determine the location of equilibria and the stability of the nonlinear dynamics systems is implemented in Maple.

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Keywords: Lotka–Volterra systems; Discrete diffusion; Routh–Hurwitz theorem; Stability; Real root isolation

1. Introduction

Since the pioneering theoretical works of Skellam [15] and Turing [20], many works have focused on the effect of spatial factors which play a crucial role in the stability of populations.

For the discrete spatial systems with diffusion, many authors considered the relationship about stability and the existence of the equilibrium between the diffusion systems and the corresponding ones without diffusion, i.e., the effect of spatial heterogeneity in patches. In identical patches case, Levin [6] showed that two unstable competitive patches can be stabilized by diffusion. Kishimoto [4] extended this result to an n-dimensional case and found [5] that a competitive system with a three-species May–Leonard heteroclinic cycle may coexist at a stable equilibrium over two patches if inter-patch migration rates are at suitable intermediate levels. Hastings [2] obtained a global stability result to a multiple species system with discrete diffusion. Takeuchi [18] proved that any Lotka–Volterra diffusion system has a nonnegative and globally stable equilibrium point under any interpatch migration rates if the isolated patches are globally stable.

Lu and Takeuchi [13] considered some two species systems and proved some sufficient condition for the systems to be permanent or globally stable. For Lotka–Volterra systems with or without diffusions [16,17], the theory of monotone flows plays a very important role in the analysis of permanence (which means that there is a compact region

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K in the interior of $R_+^m = \{x|x_i \geqslant 0;\ i = 1, \ldots, m\}$ such that all the solutions of the considered system with positive initial conditions ultimately enter and remain in K) and global stability of the system. In [19], it was shown that two Lotka–Volterra competition diffusion systems which were considered by [1] are globally stable based on the recently developed theory of numerically determining solutions of systems of polynomial equations [7]. In [13], a cooperative diffusion system is considered and the sufficient and necessary condition for the system to be permanent is obtained.

In this paper, based on the theory of monotone flows of solutions of systems of differential equations, we change the problem of the global stability and permanence for a diffusion system to be a real root isolation problem. That is to find the location and the number of the positive or nonnegative equilibria which can be solved by an algorithm of real root isolation proposed in [10,9,8]. To get the local stability of nonlinear dynamics systems, we only need to check the sign of the real parts of the eigenvalues of the characteristic equations of the linear part of the systems. The Routh–Hurwitz theorem [3] meets our requirement in checking the position of the eigenvalues in the complex plane for the considered system. Based on these, we propose an algorithm to check the local stability of the equilibria of the systems.

Section 2 contains some background concepts and fundamental results for polynomial systems and a real root isolation algorithm is described. In Section 3, we describe the Routh–Hurwitz theorem [3] and construct an algorithm to check the stability of the nonlinear dynamic systems. A cooperative system and two competitive systems are illustrated by the algorithm as examples in Sections 4 and 5, respectively.

2. Wu's well-ordering principle and real root isolation

To isolate the real zeros of a system of the polynomials, we should transform the system to a triangular form by Wu's method. We use the principle of the characteristic sets which was introduced by Ritt [14] in the context of his work on differential algebra and has been considerably developed by Wu [21,22].

Let $(PS)=\{f_1(x_1, \ldots, x_n), \ldots, f_s(x_1, \ldots, x_n)\}$ be any finite set of polynomials in n ordered variables $x_1 \prec \cdots \prec x_n$ with coefficients in certain basic field of the characteristic 0, for instance, the field of rational Q. We designate the complete set of zeros of polynomials of (PS) by $Zero(PS)$. If G is any other non-zero polynomial, the subset of $Zero(PS)$ for which $G \neq 0$ will be denoted by $Zero(PS/G)$. The following is the basic principle of the characteristic sets method [21,22].

Wu's well-ordering principle: Given a set of polynomials (PS), one can compute by an algorithmic method another set of polynomials (CS), called the characteristic set of (PS), of the triangular form

$$(CS) \qquad \begin{array}{l} c_1(u_1, \ldots, u_d;\, y_1), \\ c_2(u_1, \ldots, u_d;\, y_1, y_2), \\ \quad\vdots \\ c_r(u_1, \ldots, u_d;\, y_1, \ldots, y_r), \end{array}$$

so that

$$Zero(CS/J) \subset Zero(PS) \subset Zero(CS), \tag{1}$$

$$Zero(PS) = Zero(CS/J) \cup \bigcup_i Zero(PS_i), \tag{2}$$

where $u_1, \ldots, u_d;\, y_1, \ldots, y_r$ $(d + r = n)$ is a rearrangement of $x_1, \ldots, x_n$, $J = \prod_i I_i$, I_i is the leading coefficient of c_i as polynomial in y_i, called the initial of c_i, and $(PS_i) = (PS) \cup \{I_i\}$.

Furthermore, in proceeding with each (PS_i) like (PS) by the same principle, the zeros of PS can finally be decomposed into the union of zeros of sets of triangular polynomials in the form

$$Zero(PS) = \bigcup_i Zero(CS_i/J_i), \tag{3}$$

where (CS_i) is of triangular form as (CS) and J_i is the product of initials of polynomials in (CS_i) for each i. This is the main idea of *Ritt–Wu's zero decomposition algorithm*. In the strong form of zero decomposition theorem [22], the

CS_i can be the irreducible ascending chain. From the point of view of polynomial solving, it means that there is no multiple zeros in the triangular polynomial set. This property is a key point in our algorithm in isolating the real root of the polynomials.

In [10,8], we propose an algorithm *mrealroot* which extends the *realroot* command in Maple to the multivariate cases. The key step is to construct the maximal and minimal polynomials with respect to the original polynomials with a certain order of the variables, and to solve the constructed polynomials, we can get the real roots of the original polynomials in the forms of intervals. By Ritt–Wu's zero decomposition theorem and the algorithm of *mrealroot*, we can get all the real zeros (in the interval form) of any polynomials set *PS*.

We set the procedure of *mrealroot* in Maple with the parameters as follows:

$$mrealroot(PS, varsorder, precision, eqns).$$

Here, *PS* is the polynomial set to solve, *varsorder* is the order of the variables, precision is the calculating precision which is the biggest length of the intervals. The last parameter *eqns* is for the polynomials which need to be decided in the sign at a real root of *PS*.

Therefore, we can run the *mrealroot* command with the parameters as $mrealroot([f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)], [x_1, \ldots, x_n], c, [x_1, \ldots, x_n])$. Here, $f_i(x_1, \ldots, x_n)$ $(i = 1, \ldots, n)$ are the polynomials of the right side of the considered system. The output is the set of all the equilibria of the system. By checking the sign of $[x_1, \ldots, x_n]$, we can get the positive equilibria.

3. The Routh–Hurwitz theorem and the stability of equilibria

From the qualitative theory of ODE, we can analyze the stability of an equilibrium by considering the roots of the characteristic polynomial of the Jacobian of the corresponding linear system. If all the eigenvalues have negative real parts, the equilibrium is stable. If (at least) one of them has a positive real part, the equilibrium is unstable. When some eigenvalues have zero real parts and the remaining ones have negative real parts, the nonlinear part of the system need to be considered to decide the stability of the equilibrium.

For a high-dimensional system, it is not easy to solve the characteristic polynomial to get the exact zeros. To answer the question on stability, we only need to know whether all the eigenvalues have negative real parts or not. Therefore, the theorem of Routh and Hurwitz [3] can be used to check whether all the roots of a polynomial have negative real parts.

Routh–Hurwitz theorem: Assume that a polynomial of degree n has the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

Here, $a_0 > 0$. The determinants

$$\Delta_1 = a_1, \quad \Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \ldots,$$

$$\Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & a_{2n-4} & \cdots & a_n \end{vmatrix} \equiv a_n\Delta_{n-1}.$$

Here if $i > n$, $a_i = 0$. All the roots of the polynomial have negative real parts if and only if the following inequalities hold true:

$$a_1 > 0, \quad \Delta_2 > 0, \ldots, \Delta_{n-1} > 0, \quad a_n > 0.$$

By combining the Routh–Hurwitz theorem and the *mrealroot* algorithm, we propose an algorithm to analyze the stability of a equilibrium. The main steps of the algorithm are as follows (an algorithm to get the equilibria and check their stability):

Input: A differential system in the form of $[\dot{x}_1 = f_1(x_1, \ldots, x_n), \ldots, \dot{x}_n = f_n(x_1, \ldots, x_n)]$.

Output: The equilibria and their stability in the form.

$[[a_{i1}, b_{i1}], [a_{i2}, b_{i2}], \ldots, [a_{in}, b_{in}]]$, ‘stable’/‘unstable’.

(1) Getting the characteristic polynomial $f(\lambda)$ of the system at an equilibrium.
(2) Getting $\Delta_1, \ldots, \Delta_n$ from $f(\lambda)$ as defined in the Routh–Hurwitz theorem.
(3) Running the *mrealroot* with the arguments as

$$mrealroot([f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)], [x_1, \ldots, x_n], c, [x_1, \ldots, x_n, \Delta_1, \ldots, \Delta_n]).$$

(4) Checking the sign of the former output and obtain the stability of the equilibria.

Remarks. (i) What we need to pay attention is in the first step, we should transform the equilibrium to the origin to get characteristic polynomial. In fact, if we consider the stability of an equilibrium $(x_1^*, \ldots, x_n^*)$, each Δ_i is a polynomial in the variables of x_i^* where $(x_1^*, \ldots, x_n^*)$ is a zero of the polynomial system $[f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)]$. To use the command *mrealroot*, we need the input Δ_i as the polynomials of x_i. We deal with it in the following way and get the characteristic polynomial:
(1) Let $PS = [f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)]$.
(2) Let $\bar{x}_i = x_i - x_i^0$, and substitute it into the PS we get the PS' which is in $\bar{x}_i$ and x_i^0.
(3) Get the Jacobian of $PS' = A$ and set $f(\lambda) = \det(A - \lambda E)$, where E is the identity matrix of order n.
(4) Substitute $x_i^0 = x_i$ into $f(\lambda)$.
(ii) We have the Maple command RHconditions to get the Δ_i from a differential system. It is used in the form of *RHconditions*(PS, *vars*), in which PS is the right sides of the differential system and *vars* denotes the variables.
(iii) It is easy to check that the sign of each Δ_i from the output of *mrealroot*. And if all the Δ_i are positive, the output will be ‘stable’.

4. A cooperative system

The following system:

$$\begin{aligned}
\dot{x}_1 &= x_1(13 - 130x_1 + 31x_2) + 12(y_1 - x_1),\\
\dot{x}_2 &= x_2(13 + 531x_1 - 130x_2) + 231(y_2 - x_2),\\
\dot{y}_1 &= y_1(13 - 130y_1 + 531y_2) + 231(x_1 - y_1),\\
\dot{y}_2 &= y_2(13 + 31y_1 - 130y_2) + 12(x_2 - y_2)
\end{aligned} \tag{4}$$

is constructed in [13] to show that for a cooperative system, permanence does not imply global stability. In [13], two positive equilibria are found for system (4). By using the real root isolation algorithm, we can find all the three positive equilibria for the system.

First, we get the Δ_i $(i = 1, \ldots, 4)$ by running *RHconditions*(PS, *vars*).

$\Delta_1 = 434 - 271x_1 + 229y_1 - 271y_2 + 229x_2$, Δ_2 is a polynomial of 35 terms, Δ_3 is a polynomial of 196 terms, and Δ_4 is a polynomial of 36 terms.

Then by taking the *mrealroot* command in Maple as

$$mrealroot(F, [x_1, x_2, y_1, y_2], \tfrac{1}{1000}, [x_1, x_2, x_3, x_4, \Delta_1, \ldots, \Delta_4]),$$

where F is the set of polynomials of the right-hand side of system (4), we get the output as follows:

$$[[[0, 0], [0, 0], [0, 0], [0, 0]], [0, 0, 0, 0, +, +, -, +]],$$

$$\left[\left[\left[\tfrac{1}{10}, \tfrac{1}{10}\right], [0, 0], \left[\tfrac{1}{10}, \tfrac{1}{10}\right], [0, 0]\right], [+, 0, +, 0, +, +, -, -]\right],$$

$$[[[2, 2], [7, 7], [7, 7], [2, 2]], [+, +, +, +, +, +, +, +]],$$

$$\left[\left[[0, 0], \left[\tfrac{1}{10}, \tfrac{1}{10}\right], [0, 0], \left[\tfrac{1}{10}, \tfrac{1}{10}\right]\right], [0, +, 0, +, +, +, -, -]\right],$$

$$[[[1,1],[3,3],[3,3],[1,1]],[+,+,+,+,+,+,+,-]],$$

$$\left[\left[\left[\frac{138}{439},\frac{138}{439}\right],\left[\frac{253}{439},\frac{253}{439}\right],\left[\frac{253}{439},\frac{253}{439}\right],\left[\frac{138}{439},\frac{138}{439}\right]\right],[+,+,+,+,+,+,+,+]\right].$$

Note that although the zeros we get are in the forms of intervals, in the case of rational zeros, we can solve it precisely. This shows that there are six equilibria of the system. Three of them are positive, and the signs of Δ_i $(i=1,\ldots,4)$ for two of them are all positive. Therefore, we have:

Theorem 1. *System* (4) *has three positive equilibria*, *two of them* E_1^* *and* E_3^* *are stable and* E_2^* *is unstable*. *Here*

$$E_1^*=\left[\left[\frac{138}{439},\frac{138}{439}\right],\left[\frac{253}{439},\frac{253}{439}\right],\left[\frac{253}{439},\frac{253}{439}\right],\left[\frac{138}{439},\frac{138}{439}\right]\right],$$

$$E_2^*=[[1,1],[3,3],[3,3],[1,1]],\qquad E_3^*=[[2,2],[7,7],[7,7],[2,2]].$$

5. Two competitive systems

The following two-patch Lotka–Volterra competitive diffusion systems with different patches are proposed in [1]:

$$\begin{aligned}
\dot{x}_1&=x_1(4-x_1-y_1)+\tfrac{1}{10}(x_2-x_1),\\
\dot{x}_2&=x_2(1-x_2-y_2)+\tfrac{1}{10}(x_1-x_2),\\
\dot{y}_1&=y_1(3-2x_1-y_1)+\tfrac{1}{10}(y_2-y_1),\\
\dot{y}_2&=y_2(3-2x_2-y_2)+\tfrac{1}{10}(y_1-y_2)
\end{aligned}\tag{5}$$

and

$$\begin{aligned}
\dot{x}_1&=x_1(2-x_1-y_1)+\tfrac{1}{2}(x_2-x_1),\\
\dot{x}_2&=x_2(2-x_2-y_2)+\tfrac{1}{2}(x_1-x_2),\\
\dot{y}_1&=y_1(5-x_1-2y_1)+\tfrac{1}{2}(y_1-y_2),\\
\dot{y}_2&=y_2(\tfrac{3}{2}-x_2-2y_2)+\tfrac{1}{2}(y_2-y_1).
\end{aligned}\tag{6}$$

Computer simulations suggest in [1] that both systems have globally stable equilibria.

First, we get the Δ_i $(i=1,\ldots,4)$ of (5) by running *RHconditions*(*PS*, *vars*). Here, $\Delta_1=-106+30y_2+40x_2+30y_1+40x_1$, Δ_2 is a polynomial of 35 terms, Δ_3 is a polynomial of 196 terms and Δ_4 is a polynomial of 36 terms. Then by using *mrealroot* algorithm, we obtain 12 real roots for the right-hand side of system (5), among these 12 equilibria, 3 are boundary ones and just 1 is positive. These equilibria and the stability are as follows:

(i) The origin is unstable

$$[[[0,0],[0,0],[0,0],[0,0]],[0,0,0,0,-,-,+,+]].$$

(ii) Both boundary equilibria are unstable

$$[[[0,0],[0,0],[3,3],[3,3]],[0,0,+,+,+,+,+,-]],$$

$$\left[\left[\left[\frac{1106500486179991}{281474976710656},\frac{4426001944719965}{1125899906842624}\right],\left[\frac{1375575417195363}{1125899906842624},\frac{687787708597741}{562949953421312}\right],[0,0],[0,0]\right],\right.$$

$$\left.[+,+,0,0,+,+,+,-]\right].$$

(iii) The unique positive equilibrium is stable

$$
\begin{aligned}
E_5^* &= (x_1, x_2; y_1, y_2) \\
&= \left[\left[\frac{2170042458182231}{562949953421312}, \frac{4340084916364463}{1125899906842624}\right], \left[\frac{242971061822491}{1125899906842624}, \frac{242971090253991}{1125899906842624}\right],\right. \\
&\quad \left.\left[\frac{57227843640919}{1125899906842624}, \frac{57227853209179}{1125899906842624}\right], \left[\frac{2781484003178289}{1125899906842624}, \frac{2781484134489311}{1125899906842624}\right], +, +, +, +, +, +, +, +\right].
\end{aligned}
$$

Since the boundary equilibria in system (5) are unstable, by Theorem 1 in [19], the system is permanent. The global stability is a direct consequence of K-monotonicity for the flow together with the uniqueness of the positive equilibrium.

Theorem 2. *System* (5) *has a globally stable positive equilibrium* E_5^*.

Similarly, we can get each Δ_i $(i = 1, \dots, 4)$ of (6) by running *RHconditions*(*PS*, *vars*) so that $\Delta_1 = -85 + 50y_2 + 30x_2 + 50y_1 + 30x_1$, Δ_2 is a polynomial of 35 terms, Δ_3 is a polynomial of 196 terms and Δ_4 is a polynomial of 36 terms. We can also obtain 14 real roots for system (6), among which 3 are unstable boundary equilibria and the only positive one is

$$
\begin{aligned}
E_6^* = (x_1, x_2; y_1, y_2) = &\left[\left[\frac{8106745829}{17179869184}, \frac{4053372915}{8589934592}\right], \left[\frac{2172309209}{2147483648}, \frac{8689236847}{8589934592}\right],\right. \\
&\left.\left[\frac{36077345917}{17179869184}, \frac{36077345927}{17179869184}\right], \left[\frac{6199190141}{8589934592}, \frac{3099595075}{4294967296}\right], +, +, +, +, +, +, +, +\right].
\end{aligned}
$$

Since all the Δ_i are positive, this equilibrium is stable. By the instability of boundary equilibria and the uniqueness of the positive equilibrium, we obtain the global stability for the system.

Theorem 3. *System* (6) *has a globally stable positive equilibrium* E_6^*.

Remark. On the basis of the recently developed theory [7] of numerically determining solutions of systems of polynomial equations, 12 and 14 numerical solutions to the systems of polynomial equations of the right-hand sides of (5) and (6) have been found, respectively.

6. Discussions

In this paper, by using a real root isolation algorithm proposed in [10,8], we consider a class of Lotka–Volterra discrete diffusion systems. By using our method, all the three positive equilibria and their stability are obtained for a cooperative system, to which two positive equilibria are found in [13].

For two competitive systems proposed by Goh [1], the real root isolation algorithm can be used to check the uniqueness and the stability of a positive equilibrium. Based on the permanence by checking the sign of the Routh–Hurwitz determinants and the uniqueness of a positive equilibrium, we can get the global stability of the systems.

In all the cases, the real roots are given in the interval form with the rational end points. This algebraic solution, which is not numerical one, neither a symbolic one, can be used (as substitution) for further calculation (just as checking the sign of the Routh–Hurwitz determinants at an equilibrium). Usually, a numerical result may cause an accumulative error.

The real root isolation algorithm (for multiple variables) is also applied to construct limit cycles for two-dimensional and three-dimensional systems. An algorithmic construction for small amplitude limit cycles for more than 10 plane systems are given in [9]. A class of three-dimensional Lotka–Volterra competitive systems with multiple limit cycles are given in [11,12].

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